DEFICIENCIES OF LATTICE SUBGROUPS OF LIE GROUPS

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Abstract. Let Γ be a lattice in a connected Lie group. We show that besides a few exceptional cases, the deficiency of Γ is nonpositive.

1. Introduction

If Γ is a finitely presented group, its deficiency $\operatorname{def}(\Gamma)$ is the maximum, over all finite presentations of Γ, of the number of generators minus the number of relations. If G is a connected Lie group, a lattice in G is a discrete subgroup Γ such that G/Γ has finite volume. It is uniform if G/Γ is compact. Lubotzky proved the following result [8, Proposition 6.2]:

Theorem 1. *(Lubotzky) Let Γ be a lattice in a simple Lie group G.*
(a) If $\mathbb{R} - \operatorname{rank}(G) \geq 2$ or $G = \operatorname{Sp}(n,1)$ or $G = F_4$, then $\operatorname{def}(\Gamma) \leq 0$.
(b) If $G = \operatorname{SO}(n,1)$ (for $n \geq 3$) or $G = \operatorname{SU}(n,1)$ (for $n \geq 2$), then $\operatorname{def}(\Gamma) \leq 1$.

We give an improvement of Lubotzky's result.

Theorem 2. *Let G be a connected Lie group. Let Γ be a lattice in G. If $\operatorname{def}(\Gamma) > 0$ then*
1. Γ has a finite normal subgroup F such that Γ/F is a lattice in $\operatorname{PSL}_2(\mathbb{R})$, or
2. $\operatorname{def}(\Gamma) = 1$ and either
A. Γ is isomorphic to a torsion-free nonuniform lattice in $\mathbb{R} \times \operatorname{PSL}_2(\mathbb{R})$ or $\operatorname{PSL}_2(\mathbb{C})$, or
B. Γ is $\mathbb{Z}$, $\mathbb{Z}^2$ or the fundamental group of a Klein bottle.

The examples in case 2 do have deficiency one [5]. A free group on r generators, $r > 1$, has deficiency r and gives an example of case 1.

In some cases, we have sharper bounds on $\operatorname{def}(\Gamma)$.

Theorem 3. *1. If Γ is a lattice in $\operatorname{SO}(4,1)$ then*

$$\operatorname{def}(\Gamma) \leq 1 - \frac{3}{4\pi^2} \operatorname{vol}(H^4/\Gamma). \tag{1.1}$$

2. If Γ is a lattice in $\operatorname{SU}(2,1)$ then

$$\operatorname{def}(\Gamma) \leq 1 - \frac{6}{\pi^2} \operatorname{vol}(\mathbb{C}H^2/\Gamma). \tag{1.2}$$

(We normalize $\mathbb{C}H^2$ to have sectional curvatures between -4 and -1.)
3. If Γ is a lattice in $\operatorname{PSL}_2(\mathbb{R}) \times \operatorname{PSL}_2(\mathbb{R})$ then

$$\operatorname{def}(\Gamma) \leq 1 - \frac{1}{4\pi^2} \operatorname{vol}((H^2 \times H^2)/\Gamma). \tag{1.3}$$

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2. Proofs

To prove Theorems 2 and 3, we use methods of L^2-homology. For a review of L^2-homology, see [7]. Let G and Γ be as in the hypotheses of Theorem 2. Let $b_i^{(2)}(\Gamma) \in \mathbb{R}$ denote the i-th L^2-Betti number of Γ. Let Rad be the radical of G, let L be a Levi subgroup of G and let K be the maximal compact connected normal subgroup of L. Put $G_1 = \mathrm{Rad} \cdot K$ and $G_2 = G/G_1$, a connected semisimple Lie group whose Lie algebra has no compact factors. Let $\beta : G \to G_2$ be the projection map. Put $\Gamma_1 = \Gamma \cap G_1$ and $\Gamma_2 = \beta(\Gamma)$. Then there is an exact sequence

$$1 \longrightarrow \Gamma_1 \longrightarrow \Gamma \xrightarrow{\beta} \Gamma_2 \longrightarrow 1 \tag{2.1}$$

where Γ_1 is a lattice in G_1 and Γ_2 is a lattice in G_2 [1].

Lemma 1. *If $b_1^{(2)}(\Gamma) \neq 0$ then Γ has a finite normal subgroup F such that Γ/F is a lattice in $\mathrm{PSL}_2(\mathbb{R})$.*

Proof. There are the following possibilities:
A. Γ_1 is infinite. Then Γ has an infinite normal amenable subgroup. By a result of Cheeger and Gromov, the L^2-Betti numbers of Γ vanish [7, Theorem 10.12].
B. Γ_1 is finite and Γ_2 is finite. (That is, $\Gamma_2 = \{e\}$.) Then Γ is finite and $b_1^{(2)}(\Gamma) = 0$.
C. Γ_1 is finite and Γ_2 is infinite. By the Leray-Serre spectral sequence for L^2-homology, $b_1^{(2)}(\Gamma) = b_1^{(2)}(\Gamma_2)/|\Gamma_1|$. Suppose that $b_1^{(2)}(\Gamma_2) \neq 0$. If G_2 had an infinite center then Γ_2, being a lattice, would have to have an infinite center. This would imply by [7, Theorem 10.12] that $b_1^{(2)}(\Gamma_2)$ vanishes, so G_2 must have a finite center $Z(G_2)$. Put $G_3 = G_2/Z(G_2)$, let $\gamma : G_2 \to G_3$ be the projection and put $\Gamma_3 = \gamma(\Gamma_2)$, a lattice in G_3. Then there is the exact sequence

$$1 \longrightarrow \Gamma_2 \cap Z(G_2) \longrightarrow \Gamma_2 \xrightarrow{\gamma} \Gamma_3 \longrightarrow 1 \tag{2.2}$$

and so $b_1^{(2)}(\Gamma_2) = b_1^{(2)}(\Gamma_3)/|\Gamma_2 \cap Z(G_2)|$. Let K_3 be a maximal compact subgroup of G_3 and let $\mathcal{F}$ be a fundamental domain for the Γ_3-action on G_3/K_3. Let $\Pi(x, y)$ be the Schwartz kernel for the projection operator onto the L^2-harmonic 1-forms on G_3/K_3. By [4], $b_1^{(2)}(\Gamma_3) = \int_{\mathcal{F}} \mathrm{tr}(\Pi(x, x)) d\mathrm{vol}(x)$. Hence G_3/K_3 has nonzero L^2-harmonic 1-forms. By the Künneth formula for L^2-cohomology and [2, Section II.5], the only possibility is $G_3 = \mathrm{PSL}_2(\mathbb{R})$. Then there is the exact sequence

$$1 \longrightarrow \Gamma \cap \mathrm{Ker}(\gamma \circ \beta) \longrightarrow \Gamma \xrightarrow{\gamma \circ \beta} \Gamma_3 \longrightarrow 1 \tag{2.3}$$

with $\Gamma \cap \mathrm{Ker}(\gamma \circ \beta)$ finite. $\square$

Let geom dim Γ be the minimal dimension of a $K(\Gamma, 1)$-complex [3, p. 185]. We will need the following result of Hillman [6, Theorem 2]. For completeness, we give the short proof.

Lemma 2. *(Hillman) If Γ is a finitely-presented group then $\mathrm{def}(\Gamma) \leq 1 + b_1^{(2)}(\Gamma)$. Equality implies that there is a finite $K(\Gamma, 1)$-complex X with $\dim(X) \leq 2$.*

Proof. If Γ is finite then $\mathrm{def}(\Gamma) \leq 0$, so we may assume that Γ is infinite. Given a presentation of Γ with g generators and r relations, let X be the corresponding 2-complex. As X is two-dimensional, its second L^2-homology group is the same as the space of square-integrable

real cellular 2-cycles on the universal cover $\widetilde{X}$. This contains the ordinary integer cellular 2-cycles as a subgroup.

We have

$$\chi(X) = 1 - g + r = b_0^{(2)}(X) - b_1^{(2)}(X) + b_2^{(2)}(X) = -b_1^{(2)}(\Gamma) + b_2^{(2)}(X). \tag{2.4}$$

Hence

$$g - r = 1 + b_1^{(2)}(\Gamma) - b_2^{(2)}(X) \leq 1 + b_1^{(2)}(\Gamma). \tag{2.5}$$

If $g - r = 1 + b_1^{(2)}(\Gamma)$ then $b_2^{(2)}(X) = 0$. Hence $\mathrm{H}_2(\widetilde{X}; \mathbb{Z}) = 0$. From the Hurewicz theorem, $\widetilde{X}$ is contractible. □

We now prove Theorem 2. Suppose that $\mathrm{def}(\Gamma) > 0$. Then first of all, $|\Gamma| = \infty$. Suppose that Γ does not have a finite normal subgroup F such that G/F is a lattice in $\mathrm{PSL}_2(\mathbb{R})$. By Lemma 1, $b_1^{(2)}(\Gamma) = 0$. Then Lemma 2 implies that $\mathrm{def}(\Gamma) = 1$ and $\mathrm{geom\,dim}\,\Gamma \leq 2$. In particular, Γ is torsion-free.

As Γ_1 is a lattice in $K \cdot \mathrm{Rad}$, it is a uniform lattice [9, Chapter III]. Furthermore, as Γ_1 is a subgroup of Γ, $\mathrm{geom\,dim}\,\Gamma_1 \leq 2$ and so Γ_1 must be $\{e\}$, $\mathbb{Z}$, $\mathbb{Z}^2$ or the fundamental group of a Klein bottle. We go through the possibilities :

A. $\Gamma_1 = \{e\}$. Then $\Gamma = \Gamma_2$ is a torsion-free lattice in the semisimple group G_2. Using a result of Borel and Serre [3, p. 218], the fact that $\mathrm{geom\,dim}\Gamma \leq 2$ implies that the Lie algebra of G_2 is $\mathrm{sl}_2(\mathbb{R})$, $\mathrm{sl}_2(\mathbb{R}) \oplus \mathrm{sl}_2(\mathbb{R})$ or $\mathrm{sl}_2(\mathbb{C})$. One possibility is $G_2 = \widetilde{\mathrm{PSL}_2(\mathbb{R})}$. Using the embedding $\widetilde{\mathrm{PSL}_2(\mathbb{R})} \cong \mathbb{Z} \times_{\mathbb{Z}} \widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{PSL}_2(\mathbb{R})}$, in this case we can say that Γ is isomorphic to a lattice in $\mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{PSL}_2(\mathbb{R})}$. On the other hand, if G_2 is a finite covering of $\mathrm{PSL}_2(\mathbb{R})$ then $b_1^{(2)}(\Gamma) \neq 0$, contrary to assumption. If G_2 is an infinite covering of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ then the Leray-Serre spectral sequence implies that Γ_2 has cohomological dimension greater than two, contrary to assumption. If G_2 is a finite covering of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ then Lemma 3 below will show that $\mathrm{def}(\Gamma) \leq 0$, contrary to assumption. If $G_2 = \mathrm{SL}_2(\mathbb{C})$, let $p : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ be the projection map. Then there is the exact sequence

$$1 \longrightarrow \Gamma \cap \mathrm{Ker}(p) \longrightarrow \Gamma \xrightarrow{p} p(\Gamma) \longrightarrow 1. \tag{2.6}$$

As Γ is torsion-free, $\Gamma \cap \mathrm{Ker}(p) = \{e\}$ and so Γ is isomorphic to $p(\Gamma)$, a lattice in $\mathrm{PSL}_2(\mathbb{C})$. Thus in any case, Γ is isomorphic to a torsion-free lattice in $\mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{PSL}_2(\mathbb{R})}$ or $\mathrm{PSL}_2(\mathbb{C})$. If Γ is uniform then $\mathrm{geom\,dim}\,\Gamma = 3$. Thus Γ must be nonuniform. The torsion-free nonuniform lattices in $\mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{PSL}_2(\mathbb{R})}$ and $\mathbb{R} \times \mathrm{PSL}_2(\mathbb{R})$ are isomorphic, as they both correspond to the Seifert fiber spaces whose base is a hyperbolic orbifold with boundary [10]. We conclude that Γ is isomorphic to a torsion-free nonuniform lattice in $\mathbb{R} \times \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$.

B. $\Gamma_1 = \mathbb{Z}$. Let Γ_2' be a finite-index torsion-free subgroup of Γ_2 which acts trivially on $\mathbb{Z}$ and put $\Gamma' = \beta^{-1}(\Gamma_2')$, a finite-index subgroup of Γ. Then there is the exact sequence

$$1 \longrightarrow \Gamma_1 \longrightarrow \Gamma' \xrightarrow{\beta} \Gamma_2' \longrightarrow 1. \tag{2.7}$$

Let M be a Γ_2'-module and let $\beta^* M$ be the corresponding Γ'-module. If $\mathrm{H}^*(\Gamma_2'; M) \neq 0$, let k be the largest integer such that $\mathrm{H}^k(\Gamma_2'; M) \neq 0$. Then by the Leray-Serre spectral sequence, $\mathrm{H}^{k+1}(\Gamma'; \beta^* M) \neq 0$. As $\mathrm{geom\,dim}\,\Gamma' \leq 2$, we must have $k \leq 1$. Thus the cohomological dimension of Γ_2' is at most one and Γ_2' must be trivial or a free group [3, p. 185]. If $\Gamma_2' = \{e\}$

then $G_2 = \{e\}$ and $\Gamma = \mathbb{Z}$. If Γ_2' is a free group then G_2 is a finite covering of $\mathrm{PSL}_2(\mathbb{R})$. Let $\sigma : G_2 \to \mathrm{PSL}_2(\mathbb{R})$ be the projection map and put $L = (\sigma \circ \beta)(\Gamma)$. Then there is the exact sequence

$$1 \longrightarrow \Gamma \cap \mathrm{Ker}(\sigma \circ \beta) \longrightarrow \Gamma \xrightarrow{\sigma\circ\beta} L \longrightarrow 1 \tag{2.8}$$

where L is a lattice in $\mathrm{PSL}_2(\mathbb{R})$ and $\Gamma \cap \mathrm{Ker}(\sigma \circ \beta)$ is virtually cyclic. As $\Gamma \cap \mathrm{Ker}(\sigma \circ \beta)$ is torsion-free, it must equal $\mathbb{Z}$. It follows that Γ is isomorphic to a lattice in $\mathbb{R} \times \mathrm{PSL}_2(\mathbb{R})$ or $\mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{PSL}_2(\mathbb{R})}$. If Γ is uniform then geom dim $\Gamma = 3$. Thus Γ is nonuniform and is isomorphic to a lattice in $\mathbb{R} \times \mathrm{PSL}_2(\mathbb{R})$.
C. $\Gamma_1 = \mathbb{Z}^2$. Let Γ_2' be a finite-index torsion-free subgroup of Γ_2 which acts on $\mathbb{Z}^2$ with determinant 1 and put $\Gamma' = \beta^{-1}(\Gamma_2')$, a finite-index subgroup of Γ. Let M be a Γ_2'-module and let $\beta^* M$ be the corresponding Γ'-module. If $\mathrm{H}^*(\Gamma_2'; M) \neq 0$, let k be the largest integer such that $\mathrm{H}^k(\Gamma_2'; M) \neq 0$. Then by the Leray-Serre spectral sequence, $\mathrm{H}^{k+2}(\Gamma'; \beta^* M) \neq 0$. As geom dim $\Gamma' \leq 2$, we must have $k = 0$. Thus the cohomological dimension of Γ_2' is zero, so $\Gamma_2' = \{e\}$ and $G_2 = \{e\}$. Then $\Gamma = \mathbb{Z}^2$.
D. Γ_1 is the fundamental group of a Klein bottle. Let $\mathbb{Z}^2$ be the unique maximal abelian subgroup of Γ_1. Any automorphism of Γ_1 acts as an automorphism of $\mathbb{Z}^2$. Thus we get a homomorphism $\phi : \mathrm{Aut}(\Gamma_1) \to \mathrm{GL}_2(\mathbb{Z})$. Let $\rho : \Gamma \to \mathrm{Aut}(\Gamma_1)$ be given by $(\rho(\gamma))(\gamma_1) = \gamma\gamma_1\gamma^{-1}$. Put $\widetilde{\Gamma} = \mathrm{Ker}(\det \circ \phi \circ \rho)$, an index-2 subgroup of Γ, and put $\widetilde{\Gamma_2} = \beta\left(\widetilde{\Gamma}\right)$. Then there is an exact sequence

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \widetilde{\Gamma} \xrightarrow{\beta} \widetilde{\Gamma_2} \longrightarrow 1. \tag{2.9}$$

As in case C, it follows that $G_2 = \{e\}$ and $\Gamma = \Gamma_1$ is the fundamental group of a Klein bottle.

This proves Theorem 2. We now prove Theorem 3. Let X be as in the proof of Lemma 2. As the classifying map $X \to B\Gamma$ is 2-connected, $b_2^{(2)}(X) \geq b_2^{(2)}(\Gamma)$. Then from (2.5),

$$\mathrm{def}(\Gamma) \leq 1 + b_1^{(2)}(\Gamma) - b_2^{(2)}(\Gamma). \tag{2.10}$$

For the lattices in question, let G be the Lie group, let K now be a maximal compact subgroup of G and put $M = \Gamma \backslash G/K$, an orbifold. As G/K has no L^2-harmonic 1-forms [2, Section II.5], it follows from [4] that $b_1^{(2)}(\Gamma) = b_3^{(2)}(\Gamma) = 0$. As $|\Gamma| = \infty$, we have $b_0^{(2)}(\Gamma) = b_4^{(2)}(\Gamma) = 0$. If $\chi(\Gamma)$ is the rational-valued group Euler characteristic of Γ [3, p. 249] then

$$\chi(\Gamma) = b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(\Gamma) - b_3^{(2)}(\Gamma) + b_4^{(2)}(\Gamma) = b_2^{(2)}(\Gamma). \tag{2.11}$$

From (2.10) and (2.11), we obtain

$$\mathrm{def}(\Gamma) \leq 1 - \chi(\Gamma). \tag{2.12}$$

Furthermore, letting $e(M, g) \in \Omega^4(M)$ denote the Euler density, it follows from [4] that

$$\chi(\Gamma) = \int_M e(M, g). \tag{2.13}$$

Let G^d/K be the compact dual symmetric space to G/K. By the Hirzebruch proportionality principle,

$$\frac{\int_M e(M,g)}{\chi(G^d/K)} = \frac{\mathrm{vol}(M)}{\mathrm{vol}(G^d/K)}. \tag{2.14}$$

We have the table

G	G^d/K	$\chi(G^d/K)$	$\mathrm{vol}(G^d/K)$
$\mathrm{SO}(4,1)$	S^4	2	$\frac{8\pi^2}{3}$
$\mathrm{SU}(2,1)$	$\mathbb{C}P^2$	3	$\frac{\pi^2}{2}$
$\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$	$S^2 \times S^2$	4	$16\pi^2$.

This proves Theorem 3.

Lemma 3. *Let G be a connected Lie group with a surjective homomorphism $\rho : G \to \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ such that $\mathrm{Ker}(\rho)$ is central in G and finite. If Γ is a lattice in G then $\mathrm{def}(\Gamma) \leq 0$.*

Proof. Equation (2.12) is still valid for Γ. We have $\chi(\Gamma) = \chi(\rho(\Gamma))/|\Gamma \cap \mathrm{Ker}(\rho)|$. Applying (2.13) to $\rho(\Gamma)$, the proof of Theorem 3 gives $\chi(\rho(\Gamma)) > 0$. Hence $\chi(\Gamma) > 0$ and $\mathrm{def}(\Gamma) \leq 0$. □

References

[1] L. Auslander, "On Radicals of Discrete Subgroups of Lie Groups", Amer. J. Math. 85, p. 145-150 (1963)

[2] A. Borel and N. Wallach, Continuous Cohomology, Discrete Subgroups and Representations of Reductive Groups, Ann. of Math. Studies 94, Princeton University Press, Princeton, NJ (1980)

[3] K. Brown, Cohomology of Groups, Springer, New York (1982)

[4] J. Cheeger and M. Gromov, "Bounds on the von Neumann Dimension of L^2-Cohomology and the Gauss-Bonnet Theorem for Open Manifolds", J. Diff. Geom. 21, p. 1-34 (1985)

[5] D. Epstein, "Finite Presentations of Groups and 3-Manifolds", Quart. J. Math., Oxford, 12, p. 205-212 (1961)

[6] J. Hillman, "On L^2-Homology and Asphericity", Israel Journal of Mathematics 99, p. 271-283 (1997)

[7] W. Lück, "L^2-Invariants of Regular Coverings of Compact Manifolds and CW-Complexes (Survey)", in Handbook of Geometric Topology, Elsevier, to appear, http://wwwmath.uni-muenster.de/math/u/lueck/publ/lueck/015hand.html

[8] A. Lubotzky, "Group Presentation, p-adic Analytic Groups and Lattices in $\mathrm{SL}_2(\mathbb{C})$", Annals of Math. 118, p. 115-130 (1983)

[9] M. Raghunathan, Discrete Subgroups of Lie Groups, Springer, New York (1972)

[10] P. Scott, "The Geometry of 3-Manifolds", Bull. London Math. Soc. 15, p. 401-487 (1983)

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